115 (*Bootis* 76), *add* S. 660.

123 This is 19 *Canis Majoris*. The distance is about 10″. Given as Class II.

138 Recorded without measures as Class I. Peters in observing a minor planet found and measured a double-star which is undoubtedly identical with H. N. 138. He gives P = 331°·8; D = 4′ (*Astron. Nach.* 1635.) The place agrees closely with Herschel, and I have not been able to find any other double-star in the neighbourhood.

In the catalogue in order of Right Ascension, general number 289 (H I. 24), the North Polar Distance given as 73° 59′ 24″ should be 71° 59′ 24″; and No. 541 (H V. 11), given as 38° 44′ 3″ should be 34° 44′ 3″.

On the most Probable Result which can be derived from a number of direct Determinations of Assumed Equal Value. By E. J. Stone, M.A., F.R.S., Her Majesty's Astronomer at the Cape.

Let $x_1\, x_2 \ldots x_n$ be n direct measures of the same quantity: each apparently equally good and, by assumption, to be considered as each equally probable. Each measure is therefore, *à priori*, by assumption equally likely to be the true result; each is equally likely by the assumption to differ from the true result by an assigned quantity. Positive and negative errors therefore must be considered as equally probable to the same amount: for the greatest or least of the direct measures is, by assumption, each equally likely to be the true result.

I assume as an axiom that since all the direct measures are by assumption of equal value, or equally good, the most probable value which can be adopted is that to which each individual measure equally contributes. To obtain the most probable value, therefore, we must combine all the independent measures in such a way, that an error which may exist in one of the measures, as x_1, shall produce the same error in the "value adopted as the most probable" as would be produced by the same error in $x_2\, x_3$ or x_n.

This appears to me clear. The probable discordance of each measure from the true result is the same, and this being the case, no good reason can be assigned why we should adopt a value in which an existing error, or arbitrary change, in x_1 should produce either a greater or less error, or arbitrary change, in the adopted value than would be produced by the same error or arbitrary change, in $x_2\; x_3$ or x_n. This condition of equal contribution of the independent measures to the most probable result appears to me necessary and sufficient.

Let $u = \phi\,(x_1\, x_2 \ldots x_n)$ be the value adopted as the most probable.

Then since equal errors, or changes, in u are to be produced by the same error, or change, in x_1 x_2 . . . or x_n, we must have the following conditions satisfied amongst the partial differential coefficients,—

$$\frac{d\,u}{d\,x_1} = \frac{d\,u}{d\,x_2} \dots = \frac{d\,u}{d\,x_n}$$

$$\frac{d^2\,u}{d\,x_1^2} = \frac{d^2\,u}{d\,x_2^2} \dots = \frac{d^2\,u}{d\,x_n^2}$$

Therefore

$$\frac{d^2\,u}{d\,x_1\,d\,x_2} = \frac{d^2\,u}{d\,x_1^2} = \frac{d^2\,u}{d\,x_2^2} = \&c.$$

Hence, generally

$$\frac{d^{\,r+s}\,u}{d\,x_1^{\,r}\,.\,d\,x_2^{\,s}} = \frac{d^{\,r+s}\,u}{d\,x_1^{\,r+s}} = \frac{d^{\,r+s}\,u}{d\,x_2^{\,r+s}} = \&c.$$

Now let $x_1 = a_1 + h_1$, $x_2 = a_2 + h_2$, $x_n = a_n + h_n$. Then by comparison

$$u = \phi\,(a_1\,a_2 \dots a_n) + \left(h_1 \frac{d}{d\,x_1} + h_2 \frac{d}{d\,x_2} + \dots\right)\phi + \frac{\left(h_1 \frac{d}{d\,x_1} + \dots\right)^2 \phi}{1\,.\,2}$$

$$+ \frac{\left(h_1 \frac{d}{d\,x_1} + h_2 \frac{d}{d\,x_2} + \dots\right)^r}{\lfloor r} \phi\,(a_1 + \theta\,h_1,\; a_2 + \theta\,h_2, \dots a_n + \theta\,h_n)$$

∴ by conditions A

$$u = \phi\,(a_1\,a_2 \dots a_n) + (h_1 + h_2 \dots + h_n)\,\frac{d\,u}{d\,x_1}$$

$$+ \frac{(h_1 + h_2 \dots + h_n)^2}{1\,.\,2}\,\frac{d^2\,u}{d\,x_1^2} + \frac{(h_1 + h_2 \dots + h_n)^r}{\lfloor r}\,\frac{d^r\,\phi}{d x_1{}^r}\,(a_1 + \theta\,h_1, \dots a_n + \theta\,h_n)$$

Let

$$a_1 = a_2 \dots = a_n = s = \frac{1}{n}\,(x_1 + x_2 \dots + x_n)$$

Then

$$x_1 = s + h_1, \qquad x_2 = s + h_2, \qquad x_n = s + h_n$$

But

$$h_1 + h_2 \dots\dots + h_n = 0$$

therefore

$$u = \phi\,(s\,s \dots s) = \mathrm{F}\,(s.)$$

or u is a function of the arithmetical mean. We cannot therefore adopt a value u equally dependent upon the independent measures

$x_1\, x_2 \ldots x_n$ except it can be expressed as a function of the arithmetical mean.

But when there are only two independent measures as x_1 and x_2 we know that the most probable result is $\frac{x_1 + x_2}{2}$ since no reason can be assigned why we should adopt a value nearer to x_1 than to x_2.

$$\therefore \mathrm{F}\left(\frac{x_1 + x_2}{2}\right) = \frac{x_1 + x_2}{2} \text{ or } \mathrm{F}\,(s) = s \text{ when } n = 2.$$

Then assuming generally for n observations or measures

$$\mathrm{F}\left(\frac{x_1 + x_2 \ldots + x_n}{n}\right) = \frac{x_1 + x_2 \ldots + x_n}{n}$$

or $\mathrm{F}\,(s) = s$ for n measures.
We have

$$\mathrm{F}\left(\frac{x_1 + x_2 \;\ldots\; + x_{n+1}}{n+1}\right) = \mathrm{F}\left(\frac{n\,s + x_{n+1}}{n+1}\right)$$

$$= \mathrm{F}\left(s + \frac{x_{n+1} - s}{n+1}\right)$$

$$= \mathrm{F}\,(s) + \frac{x_{n+1} - s}{n+1}\,\mathrm{F}'\,(s) + \left(\frac{x_{n+1} - s}{n+1}\right)^2 \frac{1}{1.2}\,\mathrm{F}''\,(s) + \cdots$$

But $\mathrm{F}\,(s) = s$ by assumption, $\therefore\ \mathrm{F}'\,(s) = 1,\ \mathrm{F}''\,(s) = 0$

$$\therefore \mathrm{F}\left(\frac{x_1 + x_2 \ldots + x_n + x_{n+1}}{n+1}\right) = s + \frac{x_{n+1} - s}{n+1} = \frac{n\,s + x_{n+1}}{n+1}$$

$$\mathrm{F}\left(\frac{x_1 + x_2 \ldots + x_n + x_{n+1}}{n+1}\right) = \frac{x_1 + x_2 \ldots + x_n + x_{n+1}}{n+1}$$

$\therefore$ if the law hold for n measures it is proved for $(n + 1)$; but it has been shown to be true for $n = 2$.

$\therefore$ by successive inductions it can be shown to be generally true.

That is, the most probable result which can be deduced from the n independent direct measures $x_1\, x_2 \ldots x_n$ is the arithmetical mean provided *we assume that each of these measures is equally probable.* If we give up or change that assumption then the proposition is no longer necessarily true.

The law of frequency can of course be at once deduced from the above result.